Chp-6:Lecture Goals

• Serviceability
• Deflection calculation
• Deflection example
• Structural Design Profession is concerned with:

  Limit States Philosophy:

  Strength Limit State
  (safety-fracture, fatigue, overturning buckling etc.-)

  Serviceability
  Performance of structures under normal service load and are concerned the uses and/or occupancy of structures
The magnitudes of deflections for concrete members can be quite important. Excessive deflections of beams and slabs may cause sagging floors, ponding on flat roofs, excessive vibrations, interference with the proper operation of supported machinery, damage partitions and cause poor fitting of doors and windows, damage structure’s appearance or frighten the occupants.
\[
\Delta_{\text{max}} \text{ (at free end)} = \frac{wl^4}{8EI}
\]

\[
\Delta x = \frac{w}{24EI} \left( x^4 - 4l^3x + 3l^4 \right)
\]

\[
\Delta_{\text{max}} \text{ (at center)} = \frac{wl^4}{384EI}
\]

\[
\Delta x = \frac{wx^2}{24EI} (l - x)^2
\]
(a) Actual beam

(b) Moment diagram

(c) Cracks where $M \geq M_{cr}$

(d) Effect of cracks on effective beam cross section
\[ \lambda = \frac{\xi}{1 + 50\rho'} \]

\[ \delta_L = \delta_{D+L} - \delta_D \]

\[ \delta_{LT} = \delta_L + \lambda_\infty \delta_D + \lambda_t \delta_{SL} \]

Figure 6.4 Multipliers for long-time deflections.
(ACI Commentary Figure R9.5.2.5.)

\[ \lambda \text{ : Amplification factor} \]

\[ \zeta \text{ : Time dependent Factor} \]
The steps involved in calculating instantaneous and long-term deflections can be summarized as follows:

(a) Compute the instantaneous or short-term deflection $\delta_D$ for dead load only.
(b) Compute instantaneous deflection $\delta_{D+L}$ for dead plus full live load.
(c) Determine instantaneous deflection $\delta_L$ for full live load only.
(d) Compute instantaneous deflection due to dead load plus the sustained part of the live load $\delta_D + \delta_{SL}$.
(e) Determine instantaneous deflection $\delta_L$ for the part of the live load that is sustained.
(f) Determine the long-term deflection for dead load plus the sustained part of the live load $\delta_{LT}$.

The deflections calculated as described exceed certain limits, depending on the type of structure. Maximum deflections permitted by the ACI for several floor and roof situations were presented in Table 6.1.
<table>
<thead>
<tr>
<th>Type of member</th>
<th>Deflection to be considered</th>
<th>Deflection limitation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flat roofs not supporting or attached to nonstructural elements likely to be damaged by large deflections</td>
<td>Immediate deflection due to live load $L$</td>
<td>$\ell^*$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{180}{180}$</td>
</tr>
<tr>
<td>Floors not supporting or attached to nonstructural elements likely to be damaged by large deflections</td>
<td>Immediate deflection due to live load $L$</td>
<td>$\ell$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{360}{360}$</td>
</tr>
<tr>
<td>Roof or floor construction supporting or attached to nonstructural elements likely to be damaged by large deflections</td>
<td>That part of the total deflection occurring after attachment of nonstructural elements (sum of the long-term deflection due to all sustained loads and the immediate deflection due to any additional live load)</td>
<td>$\ell^t$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{480}{480}$</td>
</tr>
<tr>
<td>Roof or floor construction supporting or attached to nonstructural elements not likely to be damaged by large deflections</td>
<td></td>
<td>$\ell^s$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{240}{240}$</td>
</tr>
</tbody>
</table>
EXAMPLE 6.1

The beam has a simple span of 20 ft and supports a dead load including its own weight of 1 klf and a live load of 0.7 klf. $f'_c = 3000$ psi.

(a) Calculate the instantaneous deflection for $D + L$

(b) Calculate the deflection assuming that 30\% of the live load is continuously applied for three years.

Solution: (a) Instantaneous or short-term dead load deflection ($\delta_D$)

$$I_g = \frac{1}{12}(12)(20)^3 = 8000 \text{ in.}^4$$

$$M_{cr} = f_r I_g = \frac{(7.5\sqrt{3000})(8000)}{10} = 328,633 \text{ in.-lb} = 27.4 \text{ ft-k}$$

$$M_a = \frac{(1.0)(20)^2}{8} = 50 \text{ ft-k} = M_D$$
By transformed-area calculations the values of $x$ and $I_{cr}$ can be determined

$$x = 6.78''$$

$$I_{cr} = 4067 \text{ in.}^4$$

$$I_{e(midspan)} = \left( \frac{M_{cr}}{M_a} \right)^3 I_g + \left( 1 - \left( \frac{M_{cr}}{M_a} \right)^3 \right) I_{cr}$$

$$I_e = \left( \frac{27.4}{50} \right)^3 (8000) + \left[ 1 - \left( \frac{27.4}{50} \right)^3 \right] 4067 = 4714 \text{ in.}^4$$

$$E_c = 57,000 \sqrt{3000} = 3.122 \times 10^6 \text{ psi}$$

$$\delta_D = \frac{5wL^4}{384E_cI_e} = \frac{(5) \left( \frac{1000}{12} \right) (12 \times 20)^4}{(384)(3.122 \times 10^6)(4714)} = 0.245 \text{ in.}$$

**$M_a$:** Max. Service Load Moment
(b) Instantaneous or short-term deflection for dead + full live load ($\delta_{D+L}$)

\[
M_a = \frac{(1.7)(20)^2}{8} = 85 \text{ ft-k}
\]

Noting that the value of $I_e$ changes when the moments change

\[
I_e = \left(\frac{27.4}{85}\right)^3 (8000) + \left[1 - \left(\frac{27.4}{85}\right)^3\right] (4067) = 4199 \text{ in.}^4
\]

\[
\delta_{D+L} = \frac{(5)\left(\frac{1700}{12}\right)(12 \times 20)^4}{(384)(3.122 \times 10^6)(4199)} = 0.467 \text{ in.*}
\]

(c) Initial deflection for full live load ($\delta_L$)

\[
\delta_L = \delta_{D+L} - \delta_D = 0.467 - 0.245 = 0.222 \text{ in.*}
\]

This is the live load deflection that would be compared with the first or second row of Table 6.1. If the beam is part of a floor system that is "not supporting or attached to nonstructural elements likely to be damaged by large deflections" (left column of Table 6.1), then the deflection limit is $\delta_{360} = (20)(12)/360 = 0.67$ in. This limit would easily be satisfied in this case as the calculated deflection is only 0.22 in.
EXAMPLE 6.2

Determine the instantaneous deflection at the midspan of the continuous T beam shown in Figure 6.7(a). The member supports a dead load including its own weight of 1.5 k/ft and a live load of 2.5 k/ft. \( f'_c = 3000 \text{ psi} \) and \( n = 9 \). The moment diagram for full dead and live loads is shown in Figure 6.7(b), and the beam cross section is shown in Figure 6.7(c).
Figure 6.7

- Effective width = 60"
- C.G. of top steel
- Top bars at both end supports
- Bottom bars at midspan

Dimensions:
- 28"
- 23"
- 32"
- 12"
- 4"
For Positive-Moment Region

1. Locating centroidal axis for uncracked section and calculating gross moment of inertia $I_g$ and cracking moment $M_{cr}$ for the positive-moment region (Figure 6.8)

\[
\bar{y} = 10.81''
\]

\[
I_g = 60,185 \text{ in.}^4
\]

\[
M_{cr} = \frac{(7.5)(\sqrt{3000})(60,185)}{21.19} = 1,166,754 \text{ in.-lb} = 97.2 \text{ ft-k}
\]

2. Locating centroidal axis of cracked section and calculating transformed moment of inertia $I_{cr}$ for the positive-moment region (Figure 6.9)

\[
x = 5.65''
\]

\[
I_{cr} = 24,778 \text{ in.}^4
\]
3. Calculating the effective moment of inertia in the positive-moment region

\[ M_a = 150 \text{ ft-k} \]

\[ I_e = \left( \frac{97.2}{150} \right)^3 (60,185) + \left[ 1 - \left( \frac{97.2}{150} \right)^3 \right] 24,778 = 34,412 \text{ in.}^4 \]

For Negative-Moment Region

1. Locating the centroidal axis for uncracked section and calculating gross moment of inertia \( I_g \) and cracking moment \( M_{cr} \) for the negative-moment region, considering only the hatched rectangle shown in Figure 6.10

\[ \bar{y} = \left( \frac{32}{7} \right) = 16'' \]

\[ I_g = \left( \frac{12}{12} \right)(12)(32)^3 = 32,768 \text{ in.}^4 \]

\[ M_{cr} = \frac{(7.5)(\sqrt{3000})(32,768)}{16} = 841,302 \text{ in.-lb} = 70.1 \text{ ft-k} \]

The Code does not require that the designer ignore the flanges in tension for this calculation. The authors used this method to be conservative. If the tension flanges are considered, then the cracking moment is calculated from the section in Figure 6.8. The value of \( \bar{y} \) is taken to the top of the section (10.81'') because the top is in tension for negative moment, so

\[ M_{cr} = \frac{7.5\sqrt{3000}(60,185)}{10.81} = 2,287,096 \text{ in-lb} = 190.6 \text{ ft-k} \]

If this larger value for \( M_{cr} \) were used in step 3 below, the value of \( I_e \) would be 33,400 in\(^4\).
2. Locating the centroidal axis of the cracked section and calculating the transformed moment of inertia \( I_{tr} \) for the negative-moment region (Figure 6.11). See Example 2.5 for this type of calculation.

\[ x = 10.43'' \]

\[ I_{cr} = 24,147 \text{ in.}^4 \]

3. Calculating the effective moment of inertia in the negative-moment region

\[ M_a = 300 \text{ ft-k} \]

\[ I_e = \left( \frac{70.1}{300} \right)^3 (32,768) + \left[ 1 - \left( \frac{70.1}{300} \right)^3 \right] 24,147 = 24,257 \text{ in.}^4 \]
Instantaneous Deflection

The $I_e$ to be used is obtained by averaging the $I_e$ at the positive-moment section, with the average of $I_e$ computed at the negative-moment sections at the ends of the span:

$$ \text{Average } I_e = \frac{1}{2} \left[ \left( \frac{24,257 + 24,257}{2} \right) + 34,412 \right] = 29,334 \text{ in}^4 $$

$$ E_e = 57,000 \sqrt{3000} = 3.122 \times 10^6 \text{ psi} $$

Using the equation from Figure 6.2(b) and using only live loads to calculate deflections,

$$ \delta_L = \frac{w_L l^4}{384 E_e I_e} = \frac{(2.5)(30)^4}{(384)(3122)(29,334)} \cdot (1728) = 0.10 \text{ in.} $$
In this case the authors used an approximate method to calculate $\delta_L$. Instead of the cumbersome equation ($\delta_L = \delta_{D+L} - \delta_D$) we used earlier in Example 6.1(c), we simply used $w_L$ as the load in the above equation and average $I_e$. This approximation ignores the difference between $I_e$ for dead load compared with $I_e$ for dead and live load. This method gives a larger deflection, so it is conservative. Many designers have conservative approximations that they try first on many engineering calculations. If they work, there is no need to carry out the more cumbersome ones.

It has been shown that for continuous spans the Code (9.5.2.4) suggests an averaging of the $I_e$ values at the critical positive- and negative-moment sections. The ACI Commentary (R9.5.2.4) says that for approximate deflection calculations for continuous prismatic members it is satisfactory to use the midspan section properties for simple and continuous spans and at supports for cantilevers. This is because these properties, which include the effect of cracking, have the greatest effect on deflections.
Reinforced Concrete Sections - Example

Given a doubly reinforced beam with $h = 24$ in, $b = 12$ in., $d' = 2.5$ in. and $d = 21.5$ in. with 2# 7 bars in compression steel and 4# 7 bars in tension steel. The material properties are $f_c = 4$ ksi and $f_y = 60$ ksi.

Determine $I_{gt}$, $I_{cr}$, $M_{cr(+)}$, $M_{cr(-)}$, and compare to the NA of the beam.
Reinforced Concrete Sections - Example

The components of the beam

\[ A_s' = 2 \left(0.6 \text{ in}^2\right) = 1.2 \text{ in}^2 \]

\[ A_s = 4 \left(0.6 \text{ in}^2\right) = 2.4 \text{ in}^2 \]

\[ E_c = 57000 \sqrt{f_c} = 57000 \sqrt{4000} \left(\frac{1 \text{ k}}{1000 \text{ lb}}\right) \]

\[ = 3605 \text{ ksi} \]
Reinforced Concrete Sections - Example

The compute the n value and the centroid, I uncracked

\[
    n = \frac{E_s}{E_c} = \frac{29000 \text{ ksi}}{3605 \text{ ksi}} = 8.04
\]

<table>
<thead>
<tr>
<th></th>
<th>n</th>
<th>area (in²)</th>
<th>n*area (in²)</th>
<th>(y_i) (in)</th>
<th>(y_i<em>n</em>area) (in²)</th>
<th>(l) (in^4)</th>
<th>d (in)</th>
<th>(d^2<em>n</em>area) (in^6)</th>
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<tr>
<td>(A'_s)</td>
<td>7.04</td>
<td>1.2</td>
<td>8.448</td>
<td>2.5</td>
<td>21.12</td>
<td>-</td>
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<td>804.10</td>
</tr>
<tr>
<td>(A_s)</td>
<td>7.04</td>
<td>2.4</td>
<td>16.896</td>
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<td>363.26</td>
<td>-</td>
<td>9.244</td>
<td>1443.75</td>
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<tr>
<td>(A_c)</td>
<td>1</td>
<td>288</td>
<td>288</td>
<td>12</td>
<td>3456.00</td>
<td>13824</td>
<td>-0.256</td>
<td>18.89</td>
</tr>
</tbody>
</table>

\[\text{Total:}\]

\[
\begin{align*}
\text{\(313.344\)} & \quad \text{\(3840.38\)} & \quad \text{\(13824\)} & \quad \text{\(2266.74\)}
\end{align*}
\]
Reinforced Concrete Sections - Example

The compute the centroid and I uncracked

\[
\bar{y} = \frac{\sum y_i n_i A_i}{\sum n_i A_i} = \frac{3840.38 \text{ in}^3}{313.34 \text{ in}^2} = 12.26 \text{ in.}
\]

\[
I = \sum I_i + \sum d_i^2 n_i A_i = 13824 \text{ in}^4 + 2266.7 \text{ in}^4 = 16090.7 \text{ in}^4
\]
Reinforced Concrete Sections - Example

The compute the centroid and I for a cracked doubly reinforced beam.

\[
\bar{y}^2 + \frac{2(n-1)A'_s + 2nA_s}{b} \bar{y} - \frac{2(n-1)A'_s + 2nA_s d}{b} = 0
\]

\[
\bar{y}^2 + \frac{2(7.04)(1.2 \text{ in}^2) + 2(8.04)(2.4 \text{ in}^2)}{12 \text{ in.}} \bar{y} - \frac{2(7.04)(1.2 \text{ in}^2) + 2(8.04)(2.4 \text{ in}^2)(21.5 \text{ in.})}{12 \text{ in.}} = 0
\]

\[
\bar{y}^2 + 4.624 \bar{y} - 72.664 = 0
\]
Reinforced Concrete Sections - Example

The compute the centroid for a cracked doubly reinforced beam.

\[ \bar{y}^2 + 4.624 \bar{y} - 72.664 = 0 \]

\[ \bar{y} = \frac{-4.624 + \sqrt{(4.624)^2 + 4(72.664)}}{2} \]

= 6.52 in.
Reinforced Concrete Sections - Example

The compute the moment of inertia for a cracked doubly reinforced beam.

\[ I_{cr} = \frac{1}{3} b \bar{y}^3 + (n - 1) A'_s (\bar{y} - d')^2 + n A_s (d - \bar{y})^2 \]

\[ I_{cr} = \frac{1}{3} (12 \text{ in.})(6.52 \text{ in.})^3 \]

\[ + (7.04)(1.2 \text{ in}^2)(6.52 \text{ in.} - 2.5 \text{ in.})^2 \]

\[ + (8.04)(2.4 \text{ in}^2)(21.5 \text{ in.} - 6.52 \text{ in.})^2 \]

\[ = 5575.22 \text{ in}^4 \]
Reinforced Concrete Sections - Example

The critical ratio of moment of inertia

\[
\frac{I_{cr}}{I_g} = \frac{5575.22 \text{ in}^4}{16090.7 \text{ in}^4} = 0.346
\]

\[I_{cr} \approx 0.35I_g\]
Reinforced Concrete Sections - Example

Find the components of the beam

\[ C_c = 0.85 f_c ba = 0.85(4 \text{ ksi})(12 \text{ in.})(0.85) c = 34.68c \]

\[ \varepsilon'_s = \left( \frac{c - 2.5 \text{ in.}}{c} \right)(0.003) = 0.003 - \frac{0.0075}{c} \]

\[ f_s = E_s \varepsilon'_s = 29000 \left( 0.003 - \frac{0.0075}{c} \right) = 87 - \frac{217.5}{c} \]

\[ C_s = A'_s \left( f_s - 0.85 f_c \right) = \left( 1.2 \text{ in}^2 \right) \left( 87 - \frac{217.5}{c} \right) \]

\[ = 100.32 - \frac{261}{c} \]
Reinforced Concrete Sections - Example

Find the components of the beam

\[ T = \left( 2.4 \text{ in}^2 \right) \left( 60 \text{ ksi} \right) = 144 \text{ k} \]

\[ T = C_c + C_s \]

\[ 144 \text{ k} = 34.68c + 100.32 - \frac{261}{c} \implies 34.68c^2 - 43.68c - 261 = 0 \]

The neutral axis

\[ c = \frac{43.68 + \sqrt{(43.68)^2 + 4(261)(34.68)}}{2(34.68)} \]

\[ c = 3.44 \text{ in.} \]
**Reinforced Concrete Sections - Example**

The strain of the steel

\[
\varepsilon_s' = \left( \frac{3.44 \text{ in.} - 2.5 \text{ in.}}{3.44 \text{ in.}} \right) (0.003) = 0.0008 \quad 0.00207
\]

\[
\varepsilon_s = \left( \frac{21.5 \text{ in.} - 3.44 \text{ in.}}{3.44 \text{ in.}} \right) (0.003) = 0.0158 \quad 0.00207
\]

**Note:** At service loads, beams are assumed to act elastically.

\[
c = 3.44 \text{ in.}
\]

\[
\bar{y} = 6.52 \text{ in.}
\]
Reinforced Concrete Sections - Example

Using a linearly varying $\varepsilon$ and $\sigma = E\varepsilon$ along the NA is the centroid of the area for an elastic center

$$\sigma = -\frac{M_y}{I}$$

The maximum tension stress in tension is

$$f_t = 7.5\sqrt{f_c} = 7.5\sqrt{4000}$$

$$= 474.3 \text{ psi} \Rightarrow 0.4743 \text{ ksi}$$
The uncracked moments for the beam

\[
\sigma = \frac{M_y}{I} \implies M = \frac{\sigma I}{y}
\]

\[
M_{cr(+)} = \frac{f_r I}{y} = \frac{0.4743 \text{ ksi} \left(16090.7 \text{ in}^4\right)}{\left(24 \text{ in.} - 12.26 \text{ in.}\right)} = 650.2 \text{ k-in.}
\]

\[
M_{cr(-)} = \frac{f_r I}{y} = \frac{0.4743 \text{ ksi} \left(16090.7 \text{ in}^4\right)}{12.26 \text{ in.}} = 622.6 \text{ k-in.}
\]
Calculate the Deflections

(1) Instantaneous (immediate) deflections
(2) Sustained load deflection

Instantaneous Deflections

due to dead loads (unfactored), live, etc.
Calculate the Deflections

Instantaneous Deflections

Equations for calculating $\Delta_{\text{inst}}$ for common cases

\[ M_x = \frac{wx}{2} (l - x) \]
\[ \Delta_{\max} \text{ (at center)} = \frac{5wl^4}{384EI} \]
\[ \Delta_{\max} \text{ (at } x = l \sqrt{1 - \sqrt{\frac{8}{15}} = 0.5193l} \) = 0.01304 \frac{Wl^3}{EI} \]
\[ \Delta_x = \frac{wx}{180EI l^2} (3x^4 - 10l^2x^2 + 7l^4) \]
Calculate the Deflections

Instantaneous Deflections

Equations for calculating $\Delta_{\text{inst}}$ for common cases

\[ \Delta_{\text{max}} \text{ (at point of load)} \]
\[ \Delta_x \text{ (when } x \leq \frac{l}{2} \text{)} \]
\[ \Delta_{\text{max}} \left( \text{at } x = \sqrt{\frac{a(a + 2b)}{3}} \text{ when } a > b \right) \]
\[ \Delta_a \text{ (at point of load)} \]
\[ \Delta_x \text{ (when } x < a \text{)} \]
Calculate the Deflections

Instantaneous Deflections

Equations for calculating $\Delta_{\text{inst}}$ for common cases

\[
\Delta_{\text{max}} \text{ (at center)} = \frac{Pa}{24EI} (3l^2 - 4a^2)
\]

\[
\Delta x \text{ (when } x < a) = \frac{Px}{6EI} (3la - 3a^2 - x^2)
\]

\[
\Delta x \text{ (when } x > a \text{ and } < (l - a)) = \frac{Pa}{6EI} (3lx - 3x^2 - a^2)
\]

\[
\Delta_{\text{max}} \text{ (at } x = l \sqrt{\frac{1}{5}} = 0.4472l) = \frac{Pl^3}{48EI \sqrt{5}} = 0.009317 \frac{Pl^3}{EI}
\]

\[
\Delta x \text{ (at point of load)} = \frac{7Pl^3}{768EI}
\]

\[
\Delta x \text{ (when } x < \frac{l}{2}) = \frac{Px}{96EI} (3l^2 - 5x^2)
\]

\[
\Delta x \text{ (when } x > \frac{l}{2}) = \frac{P}{96EI} (x - l)^2 (11x - 2l)
\]
Calculate the Deflections

Instantaneous Deflections

Equations for calculating $\Delta_{\text{inst}}$ for common cases

1. $\Delta_{\text{max}}$ (at free end)
   \[
   \Delta_{\text{max}} = \frac{wl^4}{8EI}
   \]
   \[
   = \frac{w}{24EI} (x^4 - 4l^3x + 3l^4)
   \]

2. $\Delta_{\text{max}}$ (at center)
   \[
   \Delta_{\text{max}} = \frac{wl^4}{384EI}
   \]
   \[
   = \frac{wx^2}{24EI} (l - x)^2
   \]

3. $\Delta_{\text{max}}$ (at center)
   \[
   \Delta_{\text{max}} = \frac{pl^3}{192EI}
   \]
   \[
   = \frac{px^2}{48EI} (3l - 4x)
   \]
Sustained Load Deflections

Creep causes an increase in concrete strain \[ \Rightarrow \text{Curvature increases} \]

Compression steel present \[ \downarrow \text{Helps limit this effect.} \]

Increase in compressive strains cause increase in stress in compression reinforcement (reduces creep strain in concrete)
Sustained Load Deflections

Sustain load deflection = $\lambda \Delta_i$

$\lambda = \frac{\xi}{1 + 50 \rho'}$

ACI 9.5.2.5

$\rho' = \frac{A'_s}{bd}$ at midspan for simple and continuous beams

$\rho'$ at support for cantilever beams
**Sustained Load Deflections**

\[ \xi = \text{time dependent factor for sustained load} \]

- 5 years or more \( \Rightarrow 2.0 \)
- 12 months \( \Rightarrow 1.4 \)
- 6 months \( \Rightarrow 1.2 \)
- 3 months \( \Rightarrow 1.0 \)

Also see Figure 9.5.2.5 from ACI code
Sustained Load Deflections

For dead and live loads

\[ \Delta_{\text{total}} = \Delta_{\text{DL(\text{inst})}} + \Delta_{\text{LL(\text{inst})}} + \Delta_{\text{DL(L.T.)}} + \Delta_{\text{LL(L.T.)}} \]

DL and LL may have different \( \xi \) factors for LT (long term) \( \Delta \) calculations

\[ \Delta_{\text{total(\text{after attachment of N/S components})}} = \Delta_{\text{total}} - \Delta_{\text{DL(\text{inst})}} \]
Sustained Load Deflections

The appropriate value of $I_c$ must be used to calculate $\Delta$ at each load stage.

\[
\Delta_{DL(\text{inst})} \rightarrow \text{Some percentage of DL (if given)}
\]

\[
\Delta_{LL(\text{inst})} + \Delta_{DL(\text{inst})} \rightarrow \text{Full DL and LL}
\]
Serviceability Load Deflections - Example

Show in the attached figure is a typical interior span of a floor beam spanning between the girders at locations A and C. Partition walls, which may be damaged by large deflections, are to be erected at this level. The interior beam shown in the attached figure will support one of these partition walls. The weight of the wall is included in the uniform dead load provided in the figure. Assume that 15% of the distributed dead load is due to a superimposed dead load, which is applied to the beam after the partition wall is in place. Also assume that 40% of the live load will be sustained for at least 6 months.
Serviceability Load Deflections - Example

\[ f_c = 5 \text{ ksi} \]
\[ f_y = 60 \text{ ksi} \]
**Serviceability Load Deflections - Example**

Part I
Determine whether the floor beam meets the ACI Code maximum permissible deflection criteria. (Note: it will be assumed that it is acceptable to consider the effective moments of inertia at location A and B when computing the average effective moment of inertia for the span in this example.)

Part II
Check the ACI Code crack width provisions at midspan of the beam.
Serviceability Load Deflections - Example

Deflection before glass partition is installed (85 % of DL)

\[
b_e \leq \frac{l}{4} = \frac{35 \text{ ft} \left( 12 \text{ in/ft} \right)}{4} = 105 \text{ in}
\]

\[
\leq (8t)(2) + b_w = 8(4.5 \text{ in})2 + 12 \text{ in} = 84 \text{ in}
\]

\[
\leq s = 10 \text{ ft} = 120 \text{ in}.
\]
Serviceability Load Deflections - Example

Compute the centroid and gross moment of inertia, $I_g$.

<table>
<thead>
<tr>
<th></th>
<th>b</th>
<th>h</th>
<th>Area</th>
<th>yi</th>
<th>Ai * yi</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flange</td>
<td>84</td>
<td>4.5</td>
<td>378</td>
<td>17.75</td>
<td>6709.5</td>
</tr>
<tr>
<td>Web</td>
<td>15.5</td>
<td>12</td>
<td>186</td>
<td>7.75</td>
<td>1441.5</td>
</tr>
</tbody>
</table>

\[ \text{Flange} \]
\[ \text{Web} \]

\[ 564 \quad 8151 \]
Serviceability Load Deflections - Example

The moment of inertia

\[
\bar{y} = \frac{\sum y_i A_i}{\sum A_i} = \frac{8151 \text{ in}^3}{564 \text{ in}^2} = 14.45 \text{ in} \approx 14.5 \text{ in}
\]

\[
I_g = \sum \left( \frac{1}{12} bh^3 + Ad^2 \right)
\]

\[
= \frac{1}{12} (84 \text{ in})(4.5 \text{ in})^3 + (378 \text{ in}^2)(17.75 \text{ in} - 14.45 \text{ in})^2
\]

\[
+ \frac{1}{12} (12 \text{ in})(15.5 \text{ in})^3 + (186 \text{ in}^2)(7.75 \text{ in} - 14.45 \text{ in})^2
\]

\[
= 16,950 \text{ in}^4 \approx 16900 \text{ in}^4
\]
**Serviceability Load Deflections - Example**

The moment capacity

\[ f_r = 7.5 \sqrt{f_c} = 7.5 \sqrt{5000} = 530 \text{ psi} \]

\[ E_c = 57000 \sqrt{f_c} = 57000 \sqrt{5000} / 1000 \text{ lbs}=4030 \text{ ksi} \]

\[ M_{cr(-)} = \frac{f_r I_g}{y_{t(-)}} = \frac{(530 \text{ psi})(16900 \text{ in}^4)}{(5.55 \text{ in})(1000 \text{ lbs/kip})} = 1610 \text{ k-in} \]

\[ M_{cr(+)} = \frac{f_r I_g}{y_{t(+)}} = \frac{(530 \text{ psi})(16900 \text{ in}^4)}{(14.5 \text{ in})(1000 \text{ lbs/kip})} = 618 \text{ k-in} \]
Serviceability Load Deflections - Example

Determine bending moments due to initial load (0.85 DL) The ACI moment coefficients will be used to calculate the bending moments Since the loading is not patterned in this case, This is slightly conservative
Serviceability Load Deflections - Example

The moments at the two locations

\[ M_A = -\left(0.85w_D\right)\left(l_n\right)^2 / 11 \]
\[ = -0.85*1.00\text{k/ft}*(33.67\text{ ft})^2 / 11 \]
\[ = 87.6\text{ k-ft} \Rightarrow 1050\text{ k-in} \]

\[ M_B = (0.85w_D)\left(l_n\right)^2 / 16 \]
\[ = -0.85*1.00\text{k/ft}*(33.67\text{ ft})^2 / 16 \]
\[ = 60.2\text{ k-ft} \Rightarrow 723\text{ k-in} \]
Serviceability Load Deflections - Example

Moment at C will be set equal to $M_a$ for simplicity, as given in the problem statement.

$$M_A = 1050 \text{ k-in} < M_{cr(-)}$$

$$= 1610 \text{ k-in} \rightarrow \text{Use } I_g \text{ @ supports}$$

$$M_B = 723 \text{ k-in} > M_{cr(+)}$$

$$= 618 \text{ k-in} \rightarrow \text{Use } I_{cr} \text{ @ midspan}$$
Serviceability Load Deflections - Example

Assume Rectangular Section Behavior and calculate the areas of steel and ratio of Modulus of Elasticity

\[
n = \frac{E_s}{E_c} = \frac{29000 \text{ ksi}}{4030 \text{ ksi}} = 7.2
\]

\[
A_s' = 3 \#5 = 3 \left(0.31 \text{ in}^2\right) = 0.93 \text{ in}^2
\]

\[
A_s = 4 \#7 = 4 \left(0.6 \text{ in}^2\right) = 2.40 \text{ in}^2
\]
Serviceability Load Deflections - Example

Calculate the center of the T-beam

\[ \bar{y}^2 + \left[ \frac{2(n - 1)A_s' + 2nA_s}{b} \right] \bar{y} - \left[ \frac{2(n - 1)A_s'd' + 2nA_s d}{b} \right] = 0 \]

\[ \bar{y}^2 + \left[ \frac{2(6.2)(0.93 \text{ in}^2) + 2(7.2)(2.4 \text{ in}^2)}{84 \text{ in.}} \right] \bar{y} \]

\[ - \left[ \frac{2(6.2)(0.93 \text{ in}^2)(2.5 \text{ in.}) + 2(7.2)(2.4 \text{ in}^2)(17.5 \text{ in.})}{84 \text{ in.}} \right] = 0 \]

\[ \bar{y}^2 + 0.549\bar{y} - 7.54 = 0 \]

\[ \bar{y} = \frac{-0.549 \pm \sqrt{30.47}}{2} = 2.49 \text{ in.} \]
**Serviceability Load Deflections - Example**

The centroid is located at the \( A'_s < 4.5 \text{ in.} = t_f \). Use rectangular section behavior

\[
I_{cr(+)} = \frac{1}{3} b \bar{y}^3 + (n - 1) A'_s (\bar{y} - d')^2 + nA_s (d - \bar{y})^2
\]

\[
= \frac{1}{3} (84 \text{ in})(2.49 \text{ in})^3
\]

\[
+ (6.2)(0.93 \text{ in}^2)(2.5 \text{ in} - 2.49 \text{ in})^2
\]

\[
+ (7.2)(2.4 \text{ in}^2)(17.5 \text{ in} - 2.49 \text{ in})^2
\]

\[
= 4330 \text{ in}^4
\]
Serviceability Load Deflections - Example

The moment of inertia at midspan

\[
I_{e\text{ (midspan)}} = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left(1 - \left(\frac{M_{cr}}{M_a}\right)^3\right) I_{cr}
\]

\[
= \left(\frac{618 \text{ k-in}}{723 \text{ k-in}}\right)^3 \left(16900 \text{ in}^4\right)
\]

\[
+ \left(1 - \left(\frac{618 \text{ k-in}}{723 \text{ k-in}}\right)^3\right) \left(4330 \text{ in}^4\right)
\]

\[
= 12200 \text{ in}^4
\]
**Serviceability Load Deflections - Example**

Calculate average effective moment of inertia, $I_{e(\text{avg})}$ for interior span (for 0.85 DL) For beam with two ends continuous and use $I_g$ for the two ends.

$$
I_{e(\text{avg})} = 0.7I_{e(\text{mid})} + 0.15(I_{e1} + I_{e2})
$$

$$
= 0.7\left(12200 \text{ in}^4\right)
$$

$$
+ 0.15\left(16900 \text{ in}^4 + 16900 \text{ in}^4\right)
$$

$$
= 13600 \text{ in}^4
$$
**Serviceability Load Deflections - Example**

Calculate instantaneous deflection due to 0.85 DL:

Use the deflection equation for a fixed-fixed beam but use the span length from the centerline support to centerline support to reasonably approximate the actual deflection.

\[
\Delta_{DL \text{(inst)}} = \frac{\omega l^4}{384EI} = \frac{0.85(1.00 \text{ k/ft})(35 \text{ ft})^4(12 \text{ in/ft})^3}{384(4030 \text{ ksi})(13600 \text{ in}^4)} = 0.105 \text{ inches}
\]
Serviceability Load Deflections - Example

Calculate additional short-term Deflections (full DL & LL)

\[ M_A = \frac{-(\omega_D + \omega_L) l_n^2}{11} = \frac{-(1.62 \text{ k/ft})(33.67 \text{ ft})^2}{11} \]

\[ = -167 \text{ k-ft} = -2000 \text{ k-in} \]

\[ M_B = \frac{(\omega_D + \omega_L) l_n^2}{16} = \frac{(1.62 \text{ k/ft})(33.67 \text{ ft})^2}{16} \]

\[ = 115 \text{ k-ft} = -1380 \text{ k-in} \]
**Serviceability Load Deflections - Example**

Calculate additional short-term Deflections (full DL & LL)

Let \( M_c = M_a = -2000 \text{ k-in} \) for simplicity see problem statement

\[
M_c = M_A = |-2000 \text{ k-in}| > M_{cr(-)} = |-1610 \text{ k-in}|
\]

\( \Rightarrow \) cracking at supports

\[
M_B = 1380 \text{ k-in} > M_{cr(+)} = 618 \text{ k-in}
\]

\( \Rightarrow \) cracking at midspan
Assume beam is fully cracked under full DL + LL, therefore \( I = I_{cr} \) (do not calculate \( I_e \) for now).

\[
\begin{align*}
n &= 7.20 \\
A_s &= 2(4)(0.20 \text{ in}^2) + 3(0.6 \text{ in}^2) \\
&= 3.40 \text{ in}^2 \\
d &= 20 \text{ in} - 2.5 \text{ in} = 17.5 \text{ in} \\
A_s' &= 3(0.6 \text{ in}^2) = 1.80 \text{ in}^2 \\
d' &= 2.5 \text{ in}
\end{align*}
\]
Serviceability Load Deflections - Example

Class formula using doubly reinforced rectangular section behavior.

\[
\bar{y}^2 + \left[ \frac{\left(2(n-1)A'_s + 2nA_s\right)}{b} \right] \bar{y} - \left[ \frac{2(n-1)A'_s d' + 2nA_s d}{b} \right] = 0
\]

\[
\bar{y}^2 + \left[ \frac{2(6.2)(1.80 \text{ in}^2) + 2(7.2)(3.4 \text{ in}^2)}{12 \text{ in}} \right] \bar{y}
\]

\[
- \left[ \frac{2(6.2)(1.80 \text{ in}^2)(2.5 \text{ in}) + 2(7.2)(3.4 \text{ in}^2)(17.5 \text{ in})}{12 \text{ in}} \right] = 0
\]
Serviceability Load Deflections - Example

Class formula using doubly reinforced rectangular section behavior.

\[ \overline{y}^2 + \left[ \frac{2(n-1)A_s' + 2nA_s}{b} \right] \overline{y} - \left[ \frac{2(n-1)A_s'd' + 2nA_s d}{b} \right] = 0 \]

\[ \overline{y}^2 + 5.94\overline{y} - 76.05 = 0 \]

\[ \overline{y} = \frac{-5.94 \pm \sqrt{340}}{2} = 6.25 \text{ in.} \]
Serviceability Load Deflections - Example

Calculate moment of inertia.

\[
I_{cr(+)} = \frac{1}{3} b \bar{y}^3 + (n - 1) A'_s (\bar{y} - d')^2 + nA_s (d - \bar{y})^2
\]

\[= \frac{1}{3} (12 \text{ in})(6.25 \text{ in})^3 \]

\[+ (6.2)(1.80 \text{ in}^2)(2.5 \text{ in} - 6.25 \text{ in})^2 \]

\[+ (7.2)(3.4 \text{ in}^2)(17.5 \text{ in} - 6.25 \text{ in})^2 \]

\[= 4230 \text{ in}^4 \]
Serviceability Load Deflections - Example

Weighted $I_{cr}$

$$I_{cr} = 0.7I_{cr(mid)} + 0.15\left(I_{cr1} + I_{cr2}\right)$$

$$= 0.7\left(4330 \text{ in}^4\right) + 0.15\left(4230 \text{ in}^4 + 4230 \text{ in}^4\right)$$

$$= 4300 \text{ in}^4$$
Serviceability Load Deflections - Example

Instantaneous Dead and Live Load Deflection.

\[
\Delta_{DL(inst)} = \frac{\omega_D l^4}{384EI} = \frac{(1.00 \text{ k/ft})(35 \text{ ft})^4(12 \text{ in/ft})^3}{384(4030 \text{ ksi})(4300 \text{ in}^4)}
\]

\[= 0.390 \text{ inches}\]

\[
\Delta_{LL(inst)} = \Delta_{DL(inst)} \ast \frac{0.62 \text{ k/ft}}{1.00 \text{ k/ft}}
\]

\[= 0.242 \text{ inches}\]
Serviceability Load Deflections - Example

Long term Deflection at the midspan

\[ \rho'(t) = \frac{A_s(t)}{2b_\text{w}d} = \frac{3(0.31 \text{ in}^2)}{2(12 \text{ in})(17.5 \text{ in})} = 0.00221 \]

Dead Load (Duration > 5 years)

\[ \lambda = \frac{\xi}{1 + 50 \rho'} = \frac{2.0}{1 + 50(0.00221)} = 1.80 \]

\[ \Delta_{\text{DL(L.T.)}} = \lambda_{\text{DL}} \Delta_{\text{DL(inst)}} = 1.8(0.390 \text{ in}) = 0.702 \text{ in} \]
Serviceability Load Deflections - Example

Long term Deflection use the midspan information

\[ \rho'(t) = \frac{A_s(t)}{2b_w d} = \frac{3(0.31 \text{ in}^2)}{2(12 \text{ in})(17.5 \text{ in})} = 0.00221 \]

Live Load (40 % sustained 6 months)

\[ \lambda = \frac{\xi}{1 + 50 \rho'} = \frac{1.2}{1 + 50(0.00221)} = 1.08 \]

\[ \Delta_{LL(L.T.)} = \lambda_{LL} \Delta_{LL(\text{inst})} = 1.08(0.242 \text{ in})(0.40) = 0.105 \text{ in} \]
Serviceability Load Deflections - Example

Total Deflection after Installation of Glass Partition Wall.

\[
\Delta_{\text{total}} = \Delta_{\text{DL(inst)}} + \Delta_{\text{LL(inst)}} + \Delta_{\text{DL(L.T.)}} + \Delta_{\text{LL(L.T.)}}
\]

\[
= 0.390 \text{ in} + 0.242 \text{ in} + 0.702 \text{ in} + 0.105 \text{ in}
\]

\[
= 1.44 \text{ in}
\]

\[
\Delta_{\text{after attachment}} = \Delta_{\text{total}} - \Delta_{\text{DL(inst)}}
\]

\[
= 1.44 \text{ in} - 0.105 \text{ in} = 1.33 \text{ in}
\]

\[
\Delta_{\text{permissible}} = \frac{l}{480}
\]

\[
= \frac{35 \text{ ft}(12 \text{ in/ft})}{480} = 0.875 \text{ in} < 1.33 \text{ in (NO GOOD!)}
\]
Check whether modifying $I_{cr}$ to $I_e$ will give an acceptable deflection:

$$I_{e(midspan)} = \left( \frac{M_{cr}}{M_a} \right)^3 I_g + \left[ 1 - \left( \frac{M_{cr}}{M_a} \right)^3 \right] I_{cr}$$

$$= \left( \frac{618 \text{ k-in}}{1380 \text{ k-in}} \right)^3 (16900 \text{ in}^4)$$

$$+ \left[ 1 - \left( \frac{618 \text{ k-in}}{1380 \text{ k-in}} \right)^3 \right] (4330 \text{ in}^4)$$

$$= 5460 \text{ in}^4$$
Serviceability Load Deflections - Example

Check whether modifying $I_{cr}$ to $I_e$ will give an acceptable deflection:

$$I_{e(support)} = \left( \frac{1610 \text{ k-in}}{2000 \text{ k-in}} \right)^3 \left( 16900 \text{ in}^4 \right)$$

$$+ \left( 1 - \left( \frac{1610 \text{ k-in}}{2000 \text{ k-in}} \right)^3 \right) \left( 4230 \text{ in}^4 \right)$$

$$= 10800 \text{ in}^4$$

$$I_{e(avg)} = 0.7I_{e(mid)} + 0.15(I_{e1} + I_{e2})$$

$$= 0.7 \left( 5460 \text{ in}^4 \right) + 0.15 \left( 10800 \text{ in}^4 + 10800 \text{ in}^4 \right)$$

$$= 7060 \text{ in}^4$$
**Serviceability Load Deflections - Example**

Floor Beam meets the ACI Code Maximum permissible Deflection Criteria. Adjust deflections:

\[
\Delta_{DL\,(inst)} = 0.390 \, \text{in} \left(\frac{4300 \, \text{in}^4}{7060 \, \text{in}^4}\right) = 0.238 \, \text{in}
\]

\[
\Delta_{LL\,(inst)} = 0.242 \, \text{in} \left(\frac{4300 \, \text{in}^4}{7060 \, \text{in}^4}\right) = 0.147 \, \text{in}
\]

\[
\Delta_{DL\,(L.T.)} = 0.702 \, \text{in} \left(\frac{4300 \, \text{in}^4}{7060 \, \text{in}^4}\right) = 0.428 \, \text{in}
\]

\[
\Delta_{LL\,(L.T.)} = 0.105 \, \text{in} \left(\frac{4300 \, \text{in}^4}{7060 \, \text{in}^4}\right) = 0.064 \, \text{in}
\]
**Serviceability Load Deflections - Example**

Adjust deflections:

\[
\Delta_{\text{total}} = \Delta_{\text{DL(inst)}} + \Delta_{\text{LL(inst)}} + \Delta_{\text{DL(L.T.)}} + \Delta_{\text{LL(L.T.)}}
\]

\[
= 0.238 \text{ in.} + 0.147 \text{ in.} + 0.428 \text{ in.} + 0.064 \text{ in.}
\]

\[
= 0.877 \text{ in.}
\]

\[
\Delta_{\text{after attachment}} = \Delta_{\text{total}} - \Delta_{\text{DL(inst)}}
\]

\[
= 0.877 \text{ in.} - 0.105 \text{ in.}
\]

\[
= 0.772 \text{ in.} < \Delta_{\text{permissible}} = 0.875 \text{ in.} \quad \text{(OKAY!)}
\]
Serviceability Load Deflections - Example

Part II: Check crack width @ midspan

\[ z = f_s \sqrt[3]{d_c A} \]

\[ d_c = d_s = 2.5 \text{ in} \]

\[ A_e = 2d_s b = 2(2.5 \text{ in})(12 \text{ in}) = 60 \text{ in}^2 \]

\[ A = \frac{A_e}{\# \text{ bars}} = \frac{60 \text{ in}^2}{4} = 15 \text{ in}^2 \]
**Serviceability Load Deflections - Example**

Assume

\[ f_s = 0.6 f_y = 0.6(60 \text{ ksi}) = 36 \text{ ksi} \quad (ACI 10.6.4) \]

\[ z = (36 \text{ ksi})^{3/2}(2.5 \text{ in})(15 \text{ in}^2) \]

\[ = 120 \text{ k/in} < 175 \text{ k/in} \quad (OK) \]

For interior exposure, the crack width \( @ \) midspan is acceptable.